

Line graphs and 2-geodesic transitivity

Alice Devillers, Wei Jin*, Cai Heng Li and Cheryl E. Praeger[†]

Centre for the Mathematics of Symmetry and Computation, School of Mathematics and Statistics,
The University of Western Australia, Crawley, WA 6009, Australia

Abstract

For a graph Γ , a positive integer s and a subgroup $G \leq \text{Aut}(\Gamma)$, we prove that G is transitive on the set of s -arcs of Γ if and only if Γ has girth at least $2(s-1)$ and G is transitive on the set of $(s-1)$ -geodesics of its line graph. As applications, we first prove that the only non-complete locally cyclic 2-geodesic transitive graphs are the complete multipartite graph $K_{3[2]}$ and the icosahedron. Secondly we classify 2-geodesic transitive graphs of valency 4 and girth 3, and determine which of them are geodesic transitive.

1 Introduction

In this paper, all graphs are finite simple and undirected. An *arc* of a graph is an ordered vertex pair such that the two vertices are adjacent. A vertex triple (u, v, w) in a non-complete graph Γ with v adjacent to both u and w is a *2-arc* if $u \neq w$, and a *2-geodesic* if the distance $d_\Gamma(u, w) = 2$. A graph Γ is said to be *2-arc transitive* or *2-geodesic transitive* if its automorphism group $\text{Aut}(\Gamma)$ is transitive on arcs, and on the 2-arcs or 2-geodesics respectively. For connected graphs of girth at least 4 (where the *girth* is the length of the shortest cycle), each 2-arc is a 2-geodesic so the sets of 2-arc transitive

*The second author is supported by the Scholarships for International Research Fees (SIRF) at UWA.

[†]This paper forms part of Australian Research Council Federation Fellowship FF0776186 held by the fourth author. The first author was supported by UWA as part of the Federation Fellowship project during most of the work for this paper.

[‡]E-mail addresses: alice.devillers@uwa.edu.au (A.Devillers), 20535692@student.uwa.edu.au (W.Jin), cai.heng.li@uwa.edu.au (C.H.Li) and cheryl.praeger@uwa.edu.au (C.E.Praeger).

graphs and 2-geodesic transitive graphs are the same. However, there are also connected 2-geodesic transitive graphs of girth 3. It was shown in [7, Theorem 1.1] that for such graphs Γ , the subgraph $[\Gamma(u)]$ induced on the set $\Gamma(u)$ of vertices adjacent to u is either a connected graph of diameter 2, or is isomorphic to the disjoint union mK_r of m copies of a complete graph K_r with $m \geq 2, r \geq 2$.

One of the aims of this paper is to characterise 2-geodesic transitive graphs of girth 3 and valency 4, the smallest valency for which both possibilities for $[\Gamma(u)]$ arise, namely $[\Gamma(u)] \cong C_4$ or $2K_2$ for $u \in V(\Gamma)$. This involves the *line graph* $L(\Sigma)$ of a graph Σ , namely the graph whose vertices are the edges of Σ , with two edges adjacent in $L(\Sigma)$ if they have a vertex in common.

Theorem 1.1 *Let Γ be a finite connected non-complete graph of girth 3 and valency 4. Then Γ is 2-geodesic transitive if and only if Γ is either $L(K_4) \cong K_{3[2]}$ or $L(\Sigma)$ for a connected 3-arc transitive cubic graph Σ .*

Moreover, Γ is geodesic transitive if and only if $\Gamma = L(\Sigma)$ for a cubic distance transitive graph Σ , namely $\Sigma = K_4$, $K_{3,3}$, the Petersen graph, the Heawood graph or Tutte's 8-cage.

Since there are infinitely many 3-arc transitive cubic graphs, there are therefore infinitely many 2-geodesic transitive graphs with girth 3 and valency 4. Theorem 1.1 provides a useful method for constructing 2-geodesic transitive graphs of girth 3 and valency 4 which are not geodesic transitive, an example being the line graph of a triple cover of Tutte's 8-cage constructed in [12]. Geodesic transitivity is defined in Section 2. The line graphs mentioned in the second part of Theorem 1.1 are precisely the distance transitive graphs of valency 4 and girth 3 given, for example, in [3, Theorem 7.5.3 (i)]. For two integers $m \geq 3, b \geq 2$, $K_{m[b]}$ denotes the *complete multipartite graph* with m parts of size b .

One consequence of Theorem 1.1 is a classification of locally cyclic, 2-geodesic transitive graphs in Corollary 1.2: for $[\Gamma(u)] \cong C_n$ is connected and has diameter 2 only for valencies $n = 4$ or 5 , and the valency 5, girth 3, 2-geodesic transitive graphs were classified in [6]. We note that locally cyclic graphs are important for studying embeddings of graphs in surfaces, see for example [9, 10, 11]. We are grateful to Sandi Malnič for suggesting that we consider s -geodesic transitivity for locally cyclic graphs.

Corollary 1.2 *Let Γ be a finite connected, non-complete, locally cyclic graph. Then Γ is 2-geodesic transitive if and only if Γ is $K_{3[2]}$ or the icosahedron.*

Our second aim in the paper is to study further the disconnected case $[\Gamma(u)] \cong mK_r$ for the smallest value of m , namely $m = 2$. Each such graph

is isomorphic to the line graph of some graph, see [7, Corollary 1.5]. We investigate connections between symmetry properties of a connected graph Γ and its line graph $L(\Gamma)$. A key ingredient in this study is a collection of injective maps \mathcal{L}_s , $s \geq 1$, where \mathcal{L}_s maps the s -arcs of Γ to certain s -tuples of edges of Γ (vertices of $L(\Gamma)$) as defined in Definition 3.1. The major properties of \mathcal{L}_s are derived in Theorem 3.2 and the main consequence linking the symmetry of Γ and $L(\Gamma)$ is given in Theorem 1.3. This is given in terms of s -geodesics and s -arcs (defined in Section 2). The diameter of a graph Γ is denoted by $\text{diam}(\Gamma)$.

Theorem 1.3 *Let Γ be a finite connected regular, non-complete graph of girth g and valency at least 3. Let $G \leq \text{Aut}(\Gamma)$ and let s be a positive integer such that $2 \leq s \leq \text{diam}(L(\Gamma)) + 1$. Then G is transitive on the set of s -arcs of Γ if and only if $s \leq g/2 + 1$ and G is transitive on the set of $(s - 1)$ -geodesics of $L(\Gamma)$.*

Note that for the graph Γ and the integer s in Theorem 1.3, there is an additional restriction on s . It follows from a deep theorem of Richard Weiss in [18] that if Γ is s -arc transitive, then $s \leq 7$. This observation yields the following corollary.

Corollary 1.4 *Let Γ and g be as in Theorem 1.3. Let s be a positive integer such that $2 \leq s \leq \text{diam}(L(\Gamma)) + 1$. If $L(\Gamma)$ is $(s - 1)$ -geodesic transitive, then either $2 \leq s \leq 7$ or $s > \max\{7, g/2 + 1\}$.*

2 Preliminaries

For a graph Γ , we use $V(\Gamma)$, $E(\Gamma)$, and $\text{Aut}(\Gamma)$ to denote its *vertex set*, *edge set* and *automorphism group*, respectively. A graph Γ is said to be *vertex transitive* if the action of $\text{Aut}(\Gamma)$ on $V(\Gamma)$ is transitive. Vertex transitive graphs are regular in the sense that $|\Gamma(u)|$ is independent of $u \in V(\Gamma)$, and $|\Gamma(u)|$ is called the *valency*, denoted by $\text{val}(\Gamma)$. Regular graphs of valency 3 are called *cubic graphs*.

A subgraph X of Γ is an *induced subgraph* if two vertices of X are adjacent in X if and only if they are adjacent in Γ . For $U \subseteq V(\Gamma)$, we denote by $[U]$ the subgraph of Γ induced by U .

For two vertices u and v in $V(\Gamma)$, a *walk* from u to v is a finite sequence of vertices (v_0, v_1, \dots, v_n) such that $v_0 = u$, $v_n = v$ and $\{v_i, v_{i+1}\} \in E(\Gamma)$ for all i with $0 \leq i < n$, and n is called the *length* of the walk. If $v_i \neq v_j$ for $0 \leq i < j \leq n$, the walk is called a *path* from u to v . The smallest integer n such that there is a path of length n from u to v is called the *distance* from u

to v and is denoted by $d_\Gamma(u, v)$. The *diameter* $\text{diam}(\Gamma)$ of a connected graph Γ is the maximum of $d_\Gamma(u, v)$ over all $u, v \in V(\Gamma)$.

Let $G \leq \text{Aut}(\Gamma)$ and $s \leq \text{diam}(\Gamma)$. We say that Γ is (G, s) -*distance transitive* if, for any $t \leq s$ and for any two pairs of vertices $(u_1, v_1), (u_2, v_2)$ at distance t , there exists $g \in G$ such that $(u_1, v_1)^g = (u_2, v_2)$. If s is equal to the diameter, the graph is said to be G -*distance transitive*.

For a positive integer s , an s -*arc* of Γ is a walk (v_0, v_1, \dots, v_s) of length s such that $v_{j-1} \neq v_{j+1}$ for $1 \leq j \leq s-1$. Moreover, a 1-arc is called an arc. Suppose $G \leq \text{Aut}(\Gamma)$. Then Γ is said to be (G, s) -*arc transitive*, if Γ contains an s -arc, and for any two t -arcs α and β where $t \leq s$, there exists $g \in G$ such that $\alpha^g = \beta$. The study of (G, s) -arc transitive graphs goes back to Tutte's papers [15, 16] which showed that if Γ is a (G, s) -arc transitive cubic graph then $s \leq 5$. About twenty years later, relying on the classification of finite simple groups, Weiss [18] proved that there are no $(G, 8)$ -arc transitive graphs with valency at least three. The family of s -arc transitive graphs is a central object in algebraic graph theory, for more work see [2, 8, 13, 14, 17].

For a graph Γ and a positive integer $1 \leq s \leq \text{diam}(\Gamma)$, an s -*geodesic* of Γ is a walk (v_0, v_1, \dots, v_s) of length s such that $d_\Gamma(v_0, v_s) = s$. It is clear that 1-geodesics are arcs. For $G \leq \text{Aut}(\Gamma)$, Γ is said to be (G, s) -*geodesic transitive* if, for $1 \leq i \leq s$, G is transitive on the set of i -geodesics; further if $s = \text{diam}(\Gamma)$, then Γ is said to be G -*geodesic transitive*. Moreover, if we do not wish to specify the group we will say that Γ is s -*geodesic transitive* or *geodesic transitive* respectively, and similarly for the other properties. The study of s -geodesic transitive graphs was initiated in [6], where the properties of s -distance transitivity, s -geodesic transitivity and s -arc transitivity were compared.

A *maximum clique* of Γ is a clique with the largest possible size. The *clique graph* $C(\Gamma)$ of Γ is the graph with $V(C(\Gamma)) = \{\text{all maximum cliques of } \Gamma\}$, and two vertices are adjacent if and only if they have at least one common vertex in Γ . In particular, if Γ has girth at least 4, then $C(\Gamma)$ is the line graph $L(\Gamma)$. For example, $L(C_n) \cong C_n$ for $n \geq 3$ where C_n is the n -cycle, and $L(P_r) \cong P_{r-1}$ for $r \geq 2$ where P_r is the path with length r . The following fact about line graphs is well-known.

Lemma 2.1 [1, p.1455] *Let Γ be a connected graph. If Γ has at least 5 vertices, then $\text{Aut}(\Gamma) \cong \text{Aut}(L(\Gamma))$.*

The *subdivision graph* $S(\Gamma)$ of a graph Γ is the graph with vertex set $V(\Gamma) \cup E(\Gamma)$ and edge set $\{\{u, e\} | u \in V(\Gamma), e \in E(\Gamma), u \in e\}$. The link between the diameters of Γ and $S(\Gamma)$ was determined in [5, Remark 3.1 (b)]:

$\text{diam}(S(\Gamma)) = 2\text{diam}(\Gamma) + \delta$ for some $\delta \in \{0, 1, 2\}$. Here, based on the above result, we will show the connection between the diameters of Γ and $L(\Gamma)$ in the following lemma.

Lemma 2.2 *Let Γ be a finite connected graph with $|V(\Gamma)| \geq 2$. Then $\text{diam}(L(\Gamma)) = \text{diam}(\Gamma) + x$ for some $x \in \{-1, 0, 1\}$. Moreover, all three values occur, for example, if $\Gamma = K_{3+x}$, then $\text{diam}(L(\Gamma)) = \text{diam}(\Gamma) + x = 1 + x$ for each x .*

Proof. Let $d = \text{diam}(\Gamma)$, $d_l = \text{diam}(L(\Gamma))$ and $d_s = \text{diam}(S(\Gamma))$. Let $(x_0, x_2, \dots, x_{2d_l})$ be a d_l -geodesic of $L(\Gamma)$. Then by definition of $L(\Gamma)$, each edge intersection $x_{2i} \cap x_{2i+2}$ is a vertex x_{2i+1} of Γ and $(x_0, x_1, x_2, \dots, x_{2d_l})$ is a $2d_l$ -path in $S(\Gamma)$. Suppose that $(x_0, x_1, x_2, \dots, x_{2d_l})$ is not a $2d_l$ -geodesic of $S(\Gamma)$. Then there is an r -geodesic from x_0 to x_{2d_l} , say $(y_0, y_1, y_2, \dots, y_r)$ with $y_0 = x_0$ and $y_r = x_{2d_l}$, such that $r < 2d_l$. Since both x_0, x_{2d_l} are in $V(L(\Gamma))$, it follows that r is even, and hence $d_{L(\Gamma)}(x_0, x_{2d_l}) = \frac{r}{2} < d_l$ which contradicts that $(x_0, x_2, \dots, x_{2d_l})$ is a d_l -geodesic of $L(\Gamma)$. Thus $(x_0, x_1, x_2, \dots, x_{2d_l})$ is a $2d_l$ -geodesic in $S(\Gamma)$. It follows from [5, Remark 3.1 (b)] that $d_l \leq d_s/2 \leq d + 1$.

Now take a d_s -geodesic $(x_0, x_1, \dots, x_{d_s})$ in $S(\Gamma)$. If $x_0 \in E(\Gamma)$, then $(x_0, x_2, x_4, \dots, x_{2\lfloor d_s/2 \rfloor})$ is a $\lfloor d_s/2 \rfloor$ -geodesic in $L(\Gamma)$, so $d_l \geq \lfloor d_s/2 \rfloor \geq d$. Similarly we see that $d_l \geq d$ if $x_{d_s} \in E(\Gamma)$. Finally if both $x_0, x_{d_s} \in V(\Gamma)$, then d_s is even and $d_\Gamma(x_0, x_{d_s}) = d_s/2$. Moreover $(x_1, x_3, \dots, x_{d_s-1})$ is a $(\frac{d_s-2}{2})$ -geodesic in $L(\Gamma)$. By [5, Remark 3.1 (b)], $d_s = 2d$, so $d_l \geq \frac{d_s-2}{2} = d-1$. \square

3 Line graphs

Let Γ be a finite connected graph. For each integer $s \geq 2$, we define a map from the set of s -arcs of Γ to the set of s -tuples of $V(L(\Gamma))$.

Definition 3.1 Let $\mathbf{a} = (v_0, v_1, \dots, v_s)$ be an s -arc of Γ where $s \geq 2$, and for $0 \leq i < s$, let $e_i := \{v_i, v_{i+1}\} \in E(\Gamma)$. Define $\mathcal{L}_s(\mathbf{a}) := (e_0, e_1, \dots, e_{s-1})$.

The following theorem gives some important properties of \mathcal{L}_s .

Theorem 3.2 *Let $s \geq 2$, let Γ be a connected graph containing at least one s -arc, and let \mathcal{L}_s be as in Definition 3.1. Then the following statements hold.*

(1) \mathcal{L}_s is an injective map from the set of s -arcs of Γ to the set of $(s-1)$ -arcs of $L(\Gamma)$. Further, \mathcal{L}_s is a bijection if and only if either $s = 2$, or $s \geq 3$ and $\Gamma \cong C_m$ or P_n for some $m \geq 3, n \geq s$.

- (2) \mathcal{L}_s maps s -geodesics of Γ to $(s-1)$ -geodesics of $L(\Gamma)$.
(3) If $s \leq \text{diam}(L(\Gamma)) + 1$, then the image $\text{Im}(\mathcal{L}_s)$ contains the set \mathcal{G}_{s-1} of all $(s-1)$ -geodesics of $L(\Gamma)$. Moreover, $\text{Im}(\mathcal{L}_s) = \mathcal{G}_{s-1}$ if and only if $\text{girth}(\Gamma) \geq 2s - 2$.
(4) \mathcal{L}_s is $\text{Aut}(\Gamma)$ -equivariant, that is, $\mathcal{L}_s(\mathbf{a})^g = \mathcal{L}_s(\mathbf{a}^g)$ for all $g \in \text{Aut}(\Gamma)$ and all s -arcs \mathbf{a} of Γ .

Proof. (1) Let $\mathbf{a} = (v_0, v_1, \dots, v_s)$ be an s -arc of Γ and let $\mathcal{L}_s(\mathbf{a}) := (e_0, e_1, \dots, e_{s-1})$ with the e_i as in Definition 3.1. Then each of the e_i lies in $E(\Gamma) = V(L(\Gamma))$ and $e_k \neq e_{k+1}$ for $0 \leq k \leq s-2$. Further, since $v_j \neq v_{j+1}, v_{j+2}$ for $1 \leq j \leq s-2$, we have $e_{j-1} \neq e_{j+1}$. Thus $\mathcal{L}_s(\mathbf{a})$ is an $(s-1)$ -arc of $L(\Gamma)$.

Let $\mathbf{b} = (u_0, u_1, \dots, u_s)$ and $\mathbf{c} = (w_0, w_1, \dots, w_s)$ be two s -arcs of Γ . Then $\mathcal{L}_s(\mathbf{b}) = (f_0, f_1, \dots, f_{s-1})$ and $\mathcal{L}_s(\mathbf{c}) = (g_0, g_1, \dots, g_{s-1})$ are two $(s-1)$ -arcs of $L(\Gamma)$, where $f_i = \{u_i, u_{i+1}\}$ and $g_i = \{w_i, w_{i+1}\}$ for $0 \leq i < s$. Suppose that $\mathcal{L}_s(\mathbf{b}) = \mathcal{L}_s(\mathbf{c})$. Then $f_i = g_i$ for each $i \geq 0$, and hence $f_i \cap f_{i+1} = g_i \cap g_{i+1}$, that is, $u_{i+1} = w_{i+1}$ for each $0 \leq i \leq s-2$. So also $u_0 = w_0$ and $u_s = w_s$, and hence $\mathbf{b} = \mathbf{c}$. Thus \mathcal{L}_s is injective.

Now we prove the second part. Each arc of $L(\Gamma)$ is of the form $\mathbf{h} = (e, f)$ where $e = \{u_0, u_1\}$ and $f = \{u_1, u_2\}$ are distinct edges of Γ . Thus $u_0 \neq u_2$, so $\mathbf{k} = (u_0, u_1, u_2)$ is a 2-arc of Γ and $\mathcal{L}_2(\mathbf{k}) = \mathbf{h}$. It follows that \mathcal{L}_2 is onto and hence is a bijection. If $s \geq 3$ and $\Gamma \cong C_m$ or P_n for some $m \geq 3, n \geq s$, then $L(\Gamma) \cong C_m$ or P_{n-1} respectively, and hence for every $(s-1)$ -arc \mathbf{x} of $L(\Gamma)$, we can find an s -arc \mathbf{y} of Γ such that $\mathcal{L}_s(\mathbf{y}) = \mathbf{x}$, that is, \mathcal{L}_s is onto. Thus \mathcal{L}_s is a bijection. Conversely, suppose that \mathcal{L}_s is onto, and that $s \geq 3$. Assume that some vertex u of Γ has valency greater than 2 and let $e_1 = \{u, v_1\}, e_2 = \{u, v_2\}, e_3 = \{u, v_3\}$ be distinct edges. Then $\mathbf{x} = (e_1, e_2, e_3)$ is a 2-arc in $L(\Gamma)$ and there is no 3-arc \mathbf{y} of Γ such that $\mathcal{L}_s(\mathbf{y}) = \mathbf{x}$. In general, for $s = 3a + b \geq 4$ with $a \geq 1$ and $b \in \{0, 1, 2\}$, we concatenate a copies of \mathbf{x} to form an $(s-1)$ -arc of $L(\Gamma)$: namely (\mathbf{x}^a) if $b = 0$; (\mathbf{x}^a, e_1) if $b = 1$; (\mathbf{x}^a, e_1, e_2) if $b = 2$. This $(s-1)$ -arc does not lie in the image of \mathcal{L}_s . Thus each vertex of Γ has valency at most 2. If all vertices have valency 2 then $\Gamma \cong C_m$ for some $m \geq 3$, since Γ is connected. So suppose that some vertex u of Γ has valency 1. Since Γ is connected and each other vertex has valency at most 2, it follows that $\Gamma \cong P_n$ for some $n \geq s$.

(2) Let $\mathbf{a} = (v_0, \dots, v_s)$ be an s -geodesic of Γ and let $\mathcal{L}_s(\mathbf{a}) = (e_0, \dots, e_{s-1})$ as above. If $s = 2$, then $\mathcal{L}_s(\mathbf{a})$ is a 1-arc, and hence a 1-geodesic of $L(\Gamma)$. Suppose that $s \geq 3$ and $\mathcal{L}_s(\mathbf{a})$ is not an $(s-1)$ -geodesic. Then $d_{L(\Gamma)}(e_0, e_{s-1}) = r < s-1$ and there exists an r -geodesic $\mathbf{r} = (f_0, f_1, \dots, f_{r-1}, f_r)$ with $f_0 = e_0$ and $f_r = e_{s-1}$. Since $s \geq 3$ and \mathbf{a} is an s -geodesic, it follows that $\{v_0, v_1\} \cap \{v_{s-1}, v_s\} = \emptyset$, that is, e_0 and e_{s-1} are not adjacent in $L(\Gamma)$.

Thus $r \geq 2$. Since \mathbf{r} is an r -geodesic, it follows that the consecutive edges f_{i-1}, f_i, f_{i+1} do not share a common vertex for any $1 \leq i \leq r-1$, otherwise $(f_0, \dots, f_{i-1}, f_{i+1}, \dots, f_r)$ would be a shorter path than \mathbf{r} , which is impossible. Hence we have $f_h = \{u_h, u_{h+1}\}$ for $0 \leq h \leq r$. Then (u_1, u_2, \dots, u_r) is an $(r-1)$ -path in Γ , $\{u_1\} = e_0 \cap f_1 \subseteq \{v_0, v_1\}$ and $\{u_r\} = f_{r-1} \cap e_{s-1} \subseteq \{v_{s-1}, v_s\}$. It follows that $d_\Gamma(v_0, v_s) \leq d_\Gamma(u_1, u_r) + 2 \leq r + 1 < s$, contradicting the fact that \mathbf{a} is an s -geodesic. Therefore, $\mathcal{L}_s(\mathbf{a})$ is an $(s-1)$ -geodesic of $L(\Gamma)$.

(3) Let $2 \leq s \leq \text{diam}(L(\Gamma)) + 1$ and \mathcal{G}_{s-1} be the set of all $(s-1)$ -geodesics of $L(\Gamma)$. If $s = 2$, then by part (1), each 1-geodesic of $L(\Gamma)$ lies in the image $\text{Im}(\mathcal{L}_2)$, and hence $\mathcal{G}_1 \subseteq \text{Im}(\mathcal{L}_2)$. Now suppose inductively that $2 \leq s \leq \text{diam}(L(\Gamma))$ and $\mathcal{G}_{s-1} \subseteq \text{Im}(\mathcal{L}_s)$. Let $\mathbf{e} = (e_0, e_1, e_2, \dots, e_s)$ be an s -geodesic of $L(\Gamma)$. Then $\mathbf{e}' = (e_0, e_1, e_2, \dots, e_{s-1})$ is an $(s-1)$ -geodesic of $L(\Gamma)$. Thus there exists an s -arc \mathbf{a} of Γ such that $\mathcal{L}_s(\mathbf{a}) = \mathbf{e}'$, say $\mathbf{a} = (v_0, v_1, \dots, v_s)$. Since e_s is adjacent to $e_{s-1} = \{v_{s-1}, v_s\}$ but not to $e_{s-2} = \{v_{s-2}, v_{s-1}\}$ in $L(\Gamma)$, it follows that $e_s = \{v_s, x\}$ where $x \notin \{v_{s-2}, v_{s-1}\}$. Hence $\mathbf{b} = (v_0, v_1, \dots, v_s, x)$ is an $(s+1)$ -arc of Γ . Further, $\mathcal{L}_{s+1}(\mathbf{b}) = \mathbf{e}$. Thus $\text{Im}(\mathcal{L}_{s+1})$ contains all s -geodesics of $L(\Gamma)$, that is, $\mathcal{G}_s \subseteq \text{Im}(\mathcal{L}_{s+1})$. Hence the first part of (3) is proved by induction.

Now we prove the second part. Suppose that for every s -arc \mathbf{a} of Γ , $\mathcal{L}_s(\mathbf{a})$ is an $(s-1)$ -geodesic of $L(\Gamma)$. Let $\mathbf{g} := \text{girth}(\Gamma)$. If $s = 2$, as $\mathbf{g} \geq 3$, then $\mathbf{g} \geq 2s - 2$. Now let $s \geq 3$. Suppose that $\mathbf{g} \leq 2s - 3$. Then Γ has a \mathbf{g} -cycle $\mathbf{b} = (u_0, u_1, u_2, \dots, u_{\mathbf{g}-1}, u_{\mathbf{g}})$ with $u_{\mathbf{g}} = u_0$. It follows that $\mathcal{L}_{\mathbf{g}}(\mathbf{b})$ forms a \mathbf{g} -cycle of $L(\Gamma)$. Thus the sequence $\mathbf{b}' = (u_0, u_1, \dots, u_s)$ (where we take subscripts modulo \mathbf{g} if necessary) is an s -arc of Γ and $\mathcal{L}_s(\mathbf{b}')$ involves only the vertices of $\mathcal{L}_{\mathbf{g}}(\mathbf{b})$. This implies that $d_{L(\Gamma)}(e_0, e_{s-1}) \leq \frac{\mathbf{g}}{2} \leq \frac{2s-3}{2} < s-1$, that is, $\mathcal{L}_s(\mathbf{b}')$ is not an $(s-1)$ -geodesic, which is a contradiction. Thus, $\mathbf{g} \geq 2s - 2$.

Conversely, suppose that $\mathbf{g} \geq 2s - 2$. Let $\mathbf{a} := (v_0, v_1, v_2, \dots, v_s)$ be an s -arc of Γ . Then $\mathcal{L}_s(\mathbf{a}) = (e_0, e_1, e_2, \dots, e_{s-1})$ is an $(s-1)$ -arc of $L(\Gamma)$ by part (1). Let $\mathbf{a}' := (v_0, v_1, v_2, \dots, v_{s-1})$. Since $\mathbf{g} \geq 2s - 2$, it follows that \mathbf{a}' is an $(s-1)$ -geodesic, and hence by (2), $\mathcal{L}_{s-1}(\mathbf{a}') = (e_0, e_1, e_2, \dots, e_{s-2})$ is an $(s-2)$ -geodesic of $L(\Gamma)$. Thus $z = d_{L(\Gamma)}(e_0, e_{s-1})$ satisfies $s-3 \leq z \leq s-1$. There is a z -geodesic from e_0 to e_{s-1} , say $\mathbf{f} = (e_0, f_1, f_2, \dots, f_{z-1}, e_{s-1})$. Further, by the first part of (3), there is a $(z+1)$ -arc $\mathbf{b} = (u_0, u_1, \dots, u_z, u_{z+1})$ of Γ such that $\mathcal{L}_{z+1}(\mathbf{b}) = \mathbf{f}$ and we have $e_0 = \{u_0, u_1\} = \{v_0, v_1\}$ and $e_{s-1} = \{u_z, u_{z+1}\} = \{v_{s-1}, v_s\}$. There are 4 cases, in columns 2 and 3 of Table 1: in each case there is a given nondegenerate closed walk \mathbf{x} of length $l(\mathbf{x})$ as in Table 1. Thus $l(\mathbf{x}) \geq \mathbf{g} \geq 2s - 2$ and in each case $l(\mathbf{x}) \leq s + z - 1$. It follows that $z \geq s - 1$, and hence $z = s - 1$. Thus $\mathcal{L}_s(\mathbf{a}) = (e_0, e_1, e_2, \dots, e_{s-1})$ is an $(s-1)$ -geodesic of $L(\Gamma)$.

(4) This property follows from the definition of \mathcal{L}_s . \square

Table 1: Four cases of \mathbf{x}

Case	(u_0, u_1)	(u_z, u_{z+1})	\mathbf{x}	$l(\mathbf{x})$
1	(v_0, v_1)	(v_{s-1}, v_s)	$(v_{s-1}, v_{s-2}, \dots, v_2, v_1, u_2, \dots, u_{z-1}, v_{s-1})$	$s + z - 3$
2	(v_0, v_1)	(v_s, v_{s-1})	$(v_s, v_{s-1}, \dots, v_2, v_1, u_2, \dots, u_{z-1}, v_s)$	$s + z - 2$
3	(v_1, v_0)	(v_{s-1}, v_s)	$(v_{s-1}, v_{s-2}, \dots, v_2, v_1, u_1, u_2, \dots, u_{z-1}, v_{s-1})$	$s + z - 2$
4	(v_1, v_0)	(v_s, v_{s-1})	$(v_s, v_{s-1}, \dots, v_2, v_1, u_1, u_2, \dots, u_{z-1}, v_s)$	$s + z - 1$

Remark 3.3 (i) The map \mathcal{L}_s is usually not surjective on the set of $(s-1)$ -arcs of $L(\Gamma)$. In the proof of Theorem 3.2 (1), we constructed an $(s-1)$ -arc of $L(\Gamma)$ not in $\text{Im}(\mathcal{L}_s)$ for any Γ with at least one vertex of valency at least 3.

(ii) Theorem 3.2 (1) and (3) imply that, for each $(s-1)$ -geodesic \mathbf{e} of $L(\Gamma)$, there is a unique s -arc \mathbf{a} of Γ such that $\mathcal{L}_s(\mathbf{a}) = \mathbf{e}$. The s -arc \mathbf{a} is not always an s -geodesic. For example, if Γ has girth 3 and (v_0, v_1, v_2, v_0) is a 3-cycle, then $\mathbf{a} = (v_0, v_1, v_2)$ is not a 2-geodesic but $\mathcal{L}_2(\mathbf{a})$ is the 1-geodesic (e_0, e_1) where $e_0 = \{v_0, v_1\}$ and $e_1 = \{v_1, v_2\}$.

We are ready to prove Theorem 1.3.

Proof of Theorem 1.3. Let Γ be a connected, regular, non-complete graph of girth \mathbf{g} and valency at least 3. Then in particular $|V(\Gamma)| \geq 5$, and by Lemma 2.1, $\text{Aut}(\Gamma) \cong \text{Aut}(L(\Gamma))$. Let $G \leq \text{Aut}(\Gamma)$ and let $2 \leq s \leq \text{diam}(L(\Gamma)) + 1$.

Suppose first that G is transitive on the set of s -arcs of Γ . Then by Theorem 3.2 (4), G acts transitively on $\text{Im}(\mathcal{L}_s)$. Since $s-1 \leq \text{diam}(L(\Gamma))$, it follows that $L(\Gamma)$ has $(s-1)$ -geodesics and by Theorem 3.2 (3), $\text{Im}(\mathcal{L}_s)$ contains all the $(s-1)$ -geodesics. Thus $\text{Im}(\mathcal{L}_s)$ is the set of $(s-1)$ -geodesics of $L(\Gamma)$ and is a G -orbit. Suppose that $s > \frac{\mathbf{g}}{2} + 1$. Let $(a_0, a_1, \dots, a_{\mathbf{g}-1}, a_0)$ be a \mathbf{g} -cycle. Then $(a_0, a_1, \dots, a_{s-1}, a_s)$ is an s -arc. Since the valency of Γ is greater than 2, there exists a vertex $b (\neq a_s)$ adjacent to a_{s-1} such that $(a_0, a_1, \dots, a_{s-1}, b)$ is an s -arc. Since G is transitive on the set of s -arcs of Γ , there exists $\alpha \in G$ such that $(a_0, a_1, \dots, a_{s-1}, a_s)^\alpha = (a_0, a_1, \dots, a_{s-1}, b)$, that is, $a_s^\alpha = b$. As $a_s \in \Gamma_{\mathbf{g}-s}(a_0)$ (the set of vertices at distance $\mathbf{g}-s$ from a_0) and $a_0^\alpha = a_0$, we have $b \in \Gamma_{\mathbf{g}-s}(a_0)$. Thus there is a $(\mathbf{g}-s)$ -geodesic from a_0 to b , say $(a_0, b_1, \dots, b_{\mathbf{g}-s-1}, b_{\mathbf{g}-s} = b)$. The walk $(a_0, a_{\mathbf{g}-1}, a_{\mathbf{g}-2}, \dots, a_{s-1}, b, b_{\mathbf{g}-s-1}, \dots, b_1, a_0)$ contains a cycle with length at most $2(\mathbf{g} - (s-1))$. Since $s-1 > \frac{\mathbf{g}}{2}$, it follows that $2(\mathbf{g} - (s-1)) < \mathbf{g}$ contradicting that the girth of Γ is \mathbf{g} . Thus $s \leq \frac{\mathbf{g}}{2} + 1$.

Conversely, suppose that $s \leq \frac{\mathbf{g}}{2} + 1$ and G is transitive on the $(s-1)$ -geodesics of $L(\Gamma)$. Then by the last assertion of Theorem 3.2 (3), $\text{Im}(\mathcal{L}_s)$ is the set of $(s-1)$ -geodesics, and since \mathcal{L}_s is injective, it follows from Theorem 3.2 (1) and (4) that G is transitive on the set of s -arcs of Γ . \square

We give a brief proof of Corollary 1.4.

Proof of Corollary 1.4. Suppose that Γ, g, s are as in Theorem 1.3 and that $\text{Aut}(\Gamma)$ is transitive on the $(s - 1)$ -geodesics of $L(\Gamma)$ and $s > 7$. Then by [18], $\text{Aut}(\Gamma)$ is not transitive on the s -arcs of Γ and so by Theorem 1.3, $s > \frac{g}{2} + 1$. \square

4 Two-geodesic transitive graphs that are locally cyclic or locally $2K_2$

As discussed in Section 1, a graph Γ of valency n is locally cyclic if $[\Gamma(u)] \cong C_n$, and for such a graph to be 2-geodesic transitive (and hence in particular not a complete graph), n is 4 or 5. Also if Γ has valency 4, and Γ is 2-geodesic transitive, then $[\Gamma(u)] \cong C_4$ or $2K_2$. First we treat the case of valency 4, proving Theorem 1.1. In the proof, we will use the clique graph $C(\Gamma)$ of Γ . Recall that $C(\Gamma)$ is the graph with vertex set of all maximum cliques of Γ , and two maximum cliques are adjacent if and only if they have at least one common vertex in Γ .

Proof of Theorem 1.1. Suppose that Γ is a connected non-complete 2-geodesic transitive graph of valency 4, and let $A = \text{Aut}(\Gamma)$ and $v \in V(\Gamma)$. Then Γ is arc transitive, and so A_v is transitive on $\Gamma(v)$. If $[\Gamma(v)] \cong C_4$, then it is easy to see that $\Gamma \cong K_{3[2]}$ (or see [3, p.5] or [4]). So we may assume that $[\Gamma(v)] \cong 2K_2$. It follows from [7, Theorem 1.2] that Γ is isomorphic to the clique graph $C(\Sigma)$ of a connected graph Σ such that, for each $u \in V(\Sigma)$, the induced subgraph $[\Sigma(u)] \cong 3K_1$, that is to say, Σ is a cubic graph of girth at least 4 and $C(\Sigma)$ is in this case the line graph $L(\Sigma)$. Moreover, [7, Theorem 1.2] gives that $\Sigma \cong C(\Gamma)$. In particular, a cubic graph with girth at least 4 has $|V(\Sigma)| \geq 5$, so by Lemma 2.1, $A \cong \text{Aut}(\Sigma)$. Now we apply Theorem 1.3 to the graph Σ of girth $g \geq 4$. Since $\Gamma = L(\Sigma)$ is 2-geodesic transitive and $2 < \frac{g}{2} + 1$, it follows from Theorem 1.3 that Σ is 3-arc transitive. Therefore, Γ is the line graph of a 3-arc transitive cubic graph.

Conversely, if $\Gamma \cong K_{3[2]}$, then it is 2-geodesic transitive of girth 3. Now suppose that $\Gamma = L(\Sigma)$ where Σ is a 3-arc transitive cubic graph. If Σ had girth 3, then it would be a complete graph, which is not 3-arc transitive. Hence Σ has girth at least 4. Then Σ is locally $3K_1$, and by [7, Remark 1.2 (b)], $C(\Sigma) = L(\Sigma)$ is locally $2K_2$. Thus $L(\Sigma)$ has valency 4 and girth 3, and hence $L(\Sigma)$ is not 2-arc transitive. By Theorem 1.3 applied to Σ with $s = 2$, $L(\Sigma)$ is 2-geodesic transitive. This proves the first assertion of Theorem 1.1.

Now we suppose that Γ is geodesic transitive. Then Γ is distance transi-

tive, and so by Theorems 7.5.2 and 7.5.3 (i) of [3], Γ is one of the following graphs: $K_{3[2]} = L(K_4)$, $H(2, 3) = L(K_{3,3})$, or the line graph of the Petersen graph, the Heawood graph or Tutte's 8-cage. Further, by our argument above, $K_{3[2]}$ is geodesic transitive; by [6, Proposition 3.2], $H(2, 3)$ is geodesic transitive. It remains to consider the last three graphs.

Let Σ be the Petersen graph and $\Gamma = L(\Sigma)$. Then Σ is 3-arc transitive, and it follows from Theorem 1.3 that Γ is 2-geodesic transitive. By [3, Theorem 7.5.3 (i)], $\text{diam}(\Gamma) = 3$ and $|\Gamma(w) \cap \Gamma_3(u)| = 1$ for each 2-geodesic (u, v, w) of Γ . Thus Γ is 3-geodesic transitive, and hence is geodesic transitive.

Let Σ_1 be the Heawood graph and Σ_2 be Tutte's 8-cage. Then Σ_1 is 4-arc transitive and Σ_2 is 5-arc transitive, and hence by Theorem 1.3, $L(\Sigma_1)$ is 3-geodesic transitive and $L(\Sigma_2)$ is 4-geodesic transitive. By [3, Theorem 7.5.3 (i)], $\text{diam}(L(\Sigma_1)) = 3$ and $\text{diam}(L(\Sigma_2)) = 4$, and hence both $L(\Sigma_1)$ and $L(\Sigma_2)$ are geodesic transitive. \square

Finally, we prove Corollary 1.2.

Proof of Corollary 1.2. Let Γ be a connected non-complete locally cyclic graph of valency n . Suppose Γ is 2-geodesic transitive. Then, as discussed in the introduction, $n = 4$ or 5 . If $n = 4$, then we proved in Theorem 1.1, that $\Gamma \cong K_{3[2]}$ and that $K_{3[2]}$ is indeed 2-geodesic transitive. If $\text{val}(\Gamma) = 5$, then by [6, Theorem 1.2], Γ is isomorphic to the icosahedron, and this graph is 2-geodesic transitive. \square

References

- [1] L. Babai, Automorphism Groups, Isomorphism, Reconstruction, Handbook of Combinatorics, the Mit Press, Cambridge, Massachusetts, Amsterdam-Lausanne-New York, Vol 2, (1995), 1447–1540.
- [2] R. W. Baddeley, Two-arc transitive graphs and twisted wreath products. *J. Algebraic Combin.* **2** (1993), 215–237.
- [3] A. E. Brouwer, A. M. Cohen and A. Neumaier, Distance-Regular Graphs, Springer Ver-lag, Berlin, Heidelberg, New York, (1989).
- [4] Arjeh M. Cohen, Local recognition of graphs, buildings, and related geometries. In Finite Geometries, Buildings, and related Topics (edited by William M. Kantor, Robert A. Liebler, Stanley E. Payne, Ernest E. Shult), Oxford Sci. Publ., New York. **19** (1990), 85–94.

- [5] A. Daneshkhah, A. Devillers and C. E. Praeger, Symmetry properties of subdivision graphs, *Discrete Math.* (2011), doi:10.1016/j.disc.2011.03.031.
- [6] A. Devillers, W. Jin, C. H. Li and C. E. Praeger, On distance, geodesic and arc transitivity of graphs, preprint, 2011, available at arxiv.org/abs/1110.2235.
- [7] A. Devillers, W. Jin, C. H. Li and C. E. Praeger, Clique graphs and partial linear spaces, in preparation.
- [8] A. A. Ivanov, C. E. Praeger, On finite affine 2-arc transitive graphs. *European J. Combin.* **14** (1993), 421–444.
- [9] M. Juvan, A. Malnič and B. Mohar, Systems of curves on surfaces, *J. Combin. Theory B* **68** (1996), 7–22.
- [10] A. Malnič and B. Mohar, Generating locally cyclic triangulations of surfaces, *J. Combin. Theory B* **56** (1992), 147–164.
- [11] A. Malnič and R. Nedela, K-Minimal triangulations of surfaces, *Acta Math. Univ. Comenianae* **LXIV** **1** (1995), 57–76.
- [12] M. J. Morton, Classification of 4 and 5-arc transitive cubic graphs of small girth, *J. Austral. Math. Soc. A* **50** (1991), 138–149.
- [13] C. E. Praeger, An O’Nan Scott theorem for finite quasiprimitive permutation groups and an application to 2-arc transitive graphs, *J. London Math. Soc.* **(2)** **47** (1993), 227–239.
- [14] C. E. Praeger, On a reduction theorem for finite, bipartite, 2-arc transitive graphs, *Australas. J. Combin.* **7** (1993) 21–36.
- [15] W. T. Tutte, A family of cubical graphs, *Proc. Cambridge Philos. Soc.* **43** (1947), 459–474.
- [16] W. T. Tutte, On the symmetry of cubic graphs, *Canad. J. Math.* **11** (1959), 621–624.
- [17] R. Weiss, s-transitive graphs, *Algebraic methods in graph theory*, Vol. I, II, (Szeged, 1978), Colloq. Math. Soc. Janos Bolyai, 25, North-Holland, Amsterdam-New York, (1981), 827–847.
- [18] R. Weiss, The non-existence of 8-transitive graphs, *Combinatorica* **1** (1981), 309–311.